THE PROBLEM OF CONVERGENCE OF CONTROLLED OBJECTS

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The problem of encounter of a pursuing and persued object is investigated. A scheme for constructing the control for the pursuing object is cited. A condition is formulated under which this scheme ensures convergence of the objects not later than at a given instant.

1. Let us consider the encounter of the two controlled motions [1 to 12]

$$dy/dt = A(t)y + B(t)u \tag{1.1}$$

$$dz/dt = f(t, z, v) \tag{1.2}$$

where $y = \{y_1, \ldots, y_n\}$, $z = \{z_1, \ldots, z_n\}$ are the phase vectors of the pursuing and pursued objects, respectively; u is the r-dimensional controlling force acting on the pursuer; v is the s-dimensional control of the pursued object (target); $A(t)$ and $B(t)$ are continuous matrices of the corresponding dimensionalities; finally $f(t, z, v)$ is an n-dimensional vector function continuous in t and v which satisfies the Lipschitz condition in z.

We assume that the restrictions imposed on the control u are of the form

$$u \in U \tag{1.3}$$

where U is some convex bounded closed set in the Euclidean space E_r.

We shall not consider explicitly the character of the restrictions imposed on the control v. We merely assume that the pursuer can collide with any piecewise-continuous realization $v[t]$ from some class V,

$$v \in V \tag{1.4}$$

By the "encounter" of the motions $y[t]$ and $z[t]$ we mean the coincidence of $m \leqslant n$ prescribed components of the vectors y and z, i.e. we say that ϑ is the instant of encounter of the motions if the equalities

$$y_{i_j}[t] = z_{i_j}[t] \quad (j = 1, \ldots, m) \tag{1.5}$$

hold for the first time at $t = \vartheta$.

From now on we assume that the coordinates $i_1, \ldots, i_m$ are associated with the m-dimensional vectors $\{y\}_m$ and $\{z\}_m$.

Let ϑ° be the instant of absorption of process (1.2), (1.4) by process (1.1), (1.3) [2 and 6] computed at the initial instant $t = t_0$. We know that the problem of constructing the control $u^\circ = u^\circ[t, y, z]$ which ensures meeting of motions (1.1), (1.2) not later than at the instant ϑ° involves certain difficulties [3, 4 and 5]. Specifically, it is difficult to confine oneself to the ordinary solutions $y[t]$ and $z[t]$ of the synthesized system of differential equations (1.1), (1.2), i.e. it becomes necessary to introduce generalized motions. We shall therefore take the limit of a certain discrete scheme in which we assume that the control u_δ is constructed in the form

$$u_\delta = u_\delta[t, y[\tau_k], z[\tau_k], \quad \tau_k, \vartheta_k] \; (\tau_k \leqslant t < \tau_{k+1}, \; \tau_{k+1} - \tau_k = \delta) \qquad (1.6)$$

in each interval $[\tau_k, \tau_{k+1})$ $(k = 0, 1\ldots)$ Here ϑ_k is some ancillary variable whose meaning is explained below. (See [3 and 4] for a detailed description of the scheme.)

We say that the control $u^* = u^*[t, \; y[\tau_k], \; z[\tau_k], \; \tau_k, \; \vartheta_k]$ $(k = 0, 1\ldots)$ ensures convergence of the motions $y[t]$ and $z[t]$ form the initial state $y^\circ = y(t_0)$, $z^\circ = z(t_0)$ not later than at the instant ϑ^*, if the inequality

$$\gamma_{u^*} = \sup_{\varepsilon > 0} [\limsup_{\delta \to 0} (\sup_{v} \vartheta^\varepsilon_{u_\delta^*, v})] \leqslant \vartheta^* \qquad (1.7)$$

is fulfilled. Here $\vartheta^\varepsilon_{u_\delta^*, v}$ is the instant when for the first time $\|\{y[\vartheta] - z[\vartheta]\}_m\| \leqslant \varepsilon$.

Inequality (1.7) means that for any $\varepsilon > 0$ and for any $\Delta > 0$ there exists a $\delta^\circ < 0$ such that

$$\vartheta^\varepsilon_{u_\delta^* v} \leqslant \vartheta^* + \Delta \quad \text{for} \quad 0 < \delta \leqslant \delta^\circ, \; \cdot \; v \in V$$

The purpose of the present paper is to indicate the conditions under which one can construct a control u^* which ensures convergence of motions (1.1) and (1.2) not later than at the instant ϑ°.

2. In investigating the above problem on the encounter of motions we shall assume that condition A (formulated below) is fulfilled.

We begin by introducing some ancillary notions.

Let $G_1[y, \tau, \vartheta]$ and $G_2[z, \tau, \vartheta]$ be the domains of attainability of objects (1.1), (1.3) and (1.2), (1.4), respectively [2 and 6], in the space E_m of vectors $g = \{g_{i_1}, \ldots, g_{i_m}\}$. These domains correspond to the instant $\vartheta \geqslant \tau$ and to the initial states $y = y[\tau]$, $z = z[\tau]$.

In constructing the attainability domain $G_1[y, \tau, \vartheta]$ we assume that the measurable vector functions $u(t)$ are arbitrary and that they essentially satisfy condition (1.3) for $\tau \leqslant t \leqslant \vartheta$. The domain $G_1[y, \tau, \vartheta]$ is convex by virtue of the convexity of the set U; moreover, this domain is closed. By $\vartheta^\circ[y, z, \tau]$ we denote the instant of absorption of process (1.2), (1.4) by process (1.1), (1.3), i.e. $\vartheta^\circ[y, z, \tau]$ is the smallest value of the parameter ϑ for which $G_2[z, \tau, \vartheta] \subset G_1{}^*[y, \tau, \vartheta]$. If an instant of absorption does not exist for certain y, z, τ we stipulate that in such cases $\vartheta^\circ[y, z, \tau] = \infty$.

We say that process (1.2), (1.4) is ε-absorbed by process (1.1), (1.3) if for certain y, z, τ, ϑ we have $G_2[z, \tau, \vartheta] \subset G^\varepsilon{}_1[y, \tau, \vartheta]$, where the difference $G_1{}^\varepsilon$ is the ε-neighborhood of the set G_1 ($g \in G_1{}^\varepsilon$ if there exists a $g^* \in G_1$ such that the absolute value of the difference $\|g - g^*\| \leqslant \varepsilon$). The smallest number ε for which ε-absorption occurs will be denoted by ε° ($\varepsilon^\circ = \varepsilon^\circ[y, z, \tau, \vartheta]$).

Since the domain $G_1[y, \tau, \vartheta]$ is convex at every boundary point q of the set $G_1{}^\varepsilon[y, \tau, \vartheta]$ for $\varepsilon > 0$, we can construct one and only one hyperplane $L(q) : (l(q), g) = \mu(q)$. We shall assume that $\|l(q)\| = 1$ and that $(l(q), g) \leqslant \mu(q)$ for any $g \in G_1{}^\varepsilon[y, \tau, \vartheta]$ (i.e. that $l(q)$ determines the direction of the exterior normal to the boundary of the domain $G_1{}^\varepsilon$ at the point q). By $M_\beta(l)$ and $N_\beta(l)$ we denote the set of boundary points q of the domain $G_1{}^{\varepsilon^*}[y, \tau, \vartheta]$ satisfying the inequalities $\|l(q) - l\| \geqslant \beta$ and

$\| l(q) - l \| \leqslant \beta$, respectively, where l is a given unit vector and β is a positive number.

We define the set $\Gamma_{a,b}$ of elements $\gamma = \{y, z, \tau, \vartheta\}$ as

$$\gamma \in \Gamma_{a,b}$$

if

$$-\tau \geqslant a > 0, \ t_0 \leqslant \tau \leqslant \vartheta^\circ [y^\circ, z^\circ, t_0] = \vartheta^\circ, \ \varepsilon^\circ [y, z, \tau, \vartheta] \geqslant b > 0$$

$$y \in Y_\tau = G_1^* [y^\circ, t_0, \tau], \ z \in Z_\tau = G_2^* \ [z^\circ, t_0, \tau]$$

where a, b are positive arbitrarily small numbers, and G_1^* and G_2^* are the attainability domains of objects (1.1), (1.3) and (1.2), (1.4), respectively, constructed in the space E_n.

Condition A. There exists an $\alpha^\circ > 0$ such that for any $0 < \alpha \leqslant \alpha^\circ$ there exists a unit vector l° and a number $\beta > 0$ which satisfy the condition $\lim \beta = 0$ as $\alpha \to 0$ such that for all $q \in M_\beta (l^\circ)$ we have the inequality $\rho \{q, G_2, [z, \tau, \vartheta]\} \geqslant \alpha$. This property is fulfilled uniformly for all γ from every set $\Gamma_{a,b}$ for arbitrarily small a and b. Here $\rho(q, G_2)$ is the distance from the point q to the set G_2.

The above condition is fulfilled if for all γ from every $\Gamma_{a,b}$ the boundaries of the domains $G_1^{\varepsilon^\circ} [y, \tau, \vartheta]$ and $G_2 [z, \tau, \vartheta]$ touch at one point only, i.e. if the set

$$K [y, z, \tau, \vartheta] = D^{\varepsilon^\circ} [y, \tau, \vartheta] \cap \overline{G}_2 [z, \tau, \vartheta]$$

consists of the single point q°; here $D^{\varepsilon^\circ} [y, \tau, \vartheta]$ is the closure of the complement of the set $G_1^{\varepsilon^\circ} [y, \tau, \vartheta]$, and $\overline{G}_2 [z, \tau, \vartheta]$ is the closure of the domain $G_2 [z, \tau, \vartheta]$.

We note that $\rho\{q^\circ, G_2, [z, \tau, \vartheta]\} = 0$ for the point $q^\circ \in K [y, z, \tau, \vartheta]$. Hence, for any $0 < \alpha \leqslant \alpha^\circ$ by virtue of Condition A we have

$$q^\circ \in N_\beta (l^\circ) \tag{2.1}$$

Fig. 1 shows the case where the set K consists of the single point q°. The thick portion of the curve represents the set $N_\beta (l^\circ)$.

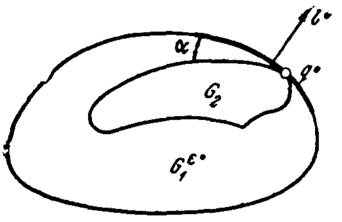

Fig. 1

Note. Let Eq. (1.2) be of the form

$$\frac{dz}{dt} = C(t) z + D(t) v \tag{2.2}$$

Here $C(t)$ and $D(t)$ are continuous matrices of the corresponding dimensionalities. For $\vartheta > \tau$ the control $v[t]$ is restricted by a condition of the form ([6], p. 71)

$$\varkappa_\tau^{(2)} \ [v] \leqslant \nu [\tau] \tag{2.3}$$

where $\varkappa_\tau^{(2)} [v]$ is the norm of the linear functional

$$\varphi_v [h] = \int_\tau^{\vartheta} (h(t), v[t]) \, dt$$

generated by the vector function $v[t]$ on the appropriate normed space $\mathscr{B}_2 \{h\}$ of the s-dimensional vector functions $h(t)$ $(\tau \leqslant t \leqslant \vartheta)$.

Let us assume that condition (1.3) can also be interpreted as the restriction $\varkappa_\tau^{(1)}[u] \leqslant \mu$ on the norm of the linear functional

$$\varphi_u [g] = \int_\tau^{\vartheta} (g(t), u[t]) \, dt$$

generated by the vector function $u[t]$ on some normed space $\mathscr{B}_1 \{g\}$ of r-dimensional vector functions $g(t)$ $(\tau \leqslant t \leqslant \vartheta)$.

Let $\rho_1 [g]$ and $\rho_2 [h]$ be the norms of the vector functions g and h in $\mathscr{B}_1 \{g\}$ and $\mathscr{B}_2 \{h\}$, respectively. In this case the instant of absorption is defined as the smallest positive root ϑ of Eq. [6]

$$\min_{\|\lambda\|\leqslant 1}\{\mu\rho_1\,[\lambda'\,\{Y\,[\vartheta,\,t]\,B\,(t)\}_m] - \nu\rho_2\,[\lambda'\{Z[\vartheta,\,t]\,D\,(t)\}_m] +$$
$$+\lambda'\{Y\,[\vartheta,\,\tau]\,y\,[\tau] - Z\,[\vartheta,\tau]\,z\,[\tau]\}_m\}=0 \qquad (2.4)$$

Here λ is an m-dimensional vector; $Y\,[\vartheta,\,t]$ and $Z\,[\vartheta,\,t]$ are the fundamental matrices of the system of Eqs.(1.1) and (2.2) which for $u\equiv 0,\ v\equiv 0$ satisfy the following condition: $Y\,[\vartheta,\,\vartheta] = E,\ Z\,[\vartheta,\vartheta] = E$; $\{F\}_m$ is the matrix whose rows are the i_1-th, i_2-th, ..., i_m-th rows of the matrix F (F is some matrix containing $n\geqslant m$ rows); the asterisk denotes transposition.

Condition A can be verified effectively in this case by means of (2.4).

The following statement is valid.

Theorem 2.1. If Condition A is fulfilled it is possible to construct a control u_δ^* of the form (1.6) which has the following property: for any arbitrarily small number $\eta > 0$ there exists a number $\delta^\circ > 0$ such that for all $0<\delta\leqslant\delta^\circ$ with the control u_δ^* chosen by the pursuer, and for all $v\in V$, there exists an instant $\vartheta\leqslant\vartheta^\circ\,[y^\circ,\,z^\circ,\,t_0]$ such that

$$\|\{y\,[\vartheta] - z\,[\vartheta]\}_m\|\leqslant\eta \qquad (2.5)$$

Thus, Theorem 2.1 states that if condition A is fulfilled there exists a control u_δ^* which ensures convergence of motions (1.1), (1.2) not later than at the instant ϑ°

Theorem 2.1 will be proved in Sections 4 and 5.

3. Let us consider the construction of the control u_δ^*. At the initial instant $t=t_0$ we determine the instant of absorption $\vartheta^\circ=\vartheta^\circ\,[y^\circ, z^\circ,\,t_0]$. We then break down the time interval $[t_0,\,\vartheta^\circ]$ into equal semi-intervals $[\tau_k,\,\tau_{k+1}),\ \tau_{k+1}-\tau_k=\delta,\ \tau_0=t_0$. At each instant $t=\tau_k$ we compute $\vartheta^\circ\,[y\,[\tau_k],\,z\,[\tau_k],\,\tau_k]$ and determine the number

$$\vartheta_k=\min\,\{\vartheta_{k-1},\,\vartheta^\circ\,[y\,[\tau_k],\,z\,[\tau_k],\,\tau_k]\},\quad \vartheta_0=\vartheta^\circ \qquad (3.1)$$

If $\vartheta_k=\vartheta^\circ\,[y\,[\tau_k],\ z\,[\tau_k],\ \tau_k]$, we construct the control $u^\circ\,(t)=u^\circ\,[t,\,y\,[\tau_k],\,z\,[\tau_k],\,\tau_k,\,\vartheta_k]$, which aims [2 and 6] the motion of system (1.1) at some point $\{y\,[\vartheta_k]\}_m=q^\circ\,[\tau_k]$ from the set $K\,[y\,[\tau_k],\,z\,[\tau_k],\,\tau_k,\,\vartheta_k]$.

Next, we set

$$u_\delta^*\,[t,\ y\,[\tau_k],\ z\,[\tau_k],\ \tau_k,\ \vartheta_k] = u^\circ\,[t,\ y\,[\tau_k],\ z\,[\tau_k],\ \tau_k,\ \vartheta_k]\quad (\tau_k\leqslant t<\tau_{k+1})$$

If $\vartheta_k<\vartheta^\circ\,[y\,[\tau_k],\,z,\,[\tau_k],\,\tau_k]$, we compute $\varepsilon^\circ\,[\tau_k], = \varepsilon^\circ\,[y\,[\tau_k],\,z\,[\tau_k],\,\tau_k,\vartheta_k]$, find some point $q^\circ\,[\tau_k]$ belonging to the set $K\,[y[\tau_k],\,z\,[\tau_k],\,\tau_k,\,\vartheta_k]$ and determine the control $u_\varepsilon^\circ\,(t)=u_\varepsilon^\circ[t,\ y\,[\tau_k],\ z\,[\tau_k],\ \tau_k,\ \vartheta_k]$, which brings system (1.1) into the $\varepsilon^\circ\,[\tau_k]$-neighborhood of the point $q^\circ[\tau_k]$. Having determined $u_\varepsilon^\circ\,(t)$, we set

$$u_\delta^*\,[t,\ y\,[\tau_k],\ z\,[\tau_k],\ \tau_k,\ \vartheta_k'] = u^\circ_\varepsilon\,[t,\ y\,[\tau_k],\ z\,[\tau_k],\ \tau_k,\ \vartheta_k]\quad (\tau_k\leqslant t<\tau_{k+1})$$

4. Before proving Theorem 2.1 we consider the following ancillary problem.

Problem 4.1. Let the motion of an object be described by Eq.(1.1) where the control is restricted by a condition of the form (1.3). We assume that the domain $G_1^\varepsilon\,[y,\,\tau,\,\vartheta]$ has been constructed for certain values $\varepsilon\geqslant b>0,\ \vartheta>\tau,\ y=y\,[\tau]$. Let q_1 and q_2 be certain boundary points of the domain G_1^ε such that

$$\|\,l\,(q_1) - l\,(q_2)\| = \varphi \quad \text{(is some small parameter)} \qquad (4.1)$$

By $u_1\,(t)$ and $u_2\,(t)$ we denote the permissible program controls which bring system (1.1) from the state $y\,[\tau]$ to $y_1\,[\vartheta]$ and $y_2\,[\vartheta]$, respectively, such that

$$\|\{y_1[\vartheta]\}_m - q_1\| = \varepsilon, \qquad \|\{y_2[\vartheta]\}_m - q_2\| = \varepsilon \tag{4.2}$$

We assume that in the time interval $[\tau, \tau+\delta]$, $\tau+\delta < \vartheta$ system (1.1) is acted on by the control $u_2(t)$ which produces the motion $y_2[t]$. If we construct the domain $G_1^\varepsilon[y_2[\tau+\delta], \tau+\delta, \vartheta]$ from the value of $y_2[\tau+\delta]$ realized at the instant $t = \tau+\delta$, then, generally speaking, $q_1 \overline{\in} G_1^\varepsilon[y_2[\tau+\delta], \tau+\delta, \vartheta]$. We must choose ε^* in such a way that $q_1 \in G_1^{\varepsilon^*}[y_2[\tau+\delta], \tau+\delta, \vartheta]$ and estimate the quantity $\Delta\varepsilon = \varepsilon^* - \varepsilon$.

Solution of Problem 4.1. By the Cauchy formula we have

$$\begin{aligned} y_1[\vartheta] &= Y[\vartheta, \tau]\, y[\tau] + \int_\tau^\vartheta Y[\vartheta, t]\, B(t)\, u_1(t)\, dt \\ y_2[\vartheta] &= Y[\vartheta, \tau]\, y[\tau] + \int_\tau^\vartheta Y[\vartheta, t]\, B(t)\, u_2(t)\, dt \end{aligned} \tag{4.3}$$

We introduce the following notation:

$$\begin{aligned} &\Delta u(t) = u_1(t) - u_2(t) \\ &u_1^*(t) = \begin{cases} u_1(t) - \Delta u(t) = u_2(t) & (\tau \leqslant t < \tau+\delta) \\ u_1(t) & (\tau+\delta \leqslant t \leqslant \vartheta) \end{cases} \\ &u_2^*(t) = \begin{cases} u_2(t) + \Delta u(t) = u_1(t) & (\tau \leqslant t < \tau+\delta) \\ u_2(t) & (\tau+\delta \leqslant t \leqslant \vartheta) \end{cases} \end{aligned} \tag{4.4}$$

The controls $u_1^*(t)$ and $u_2^*(t)$ are permissible and are associated with certain trajectories $y_1^*[t]$ and $y_2^*[t]$. From (4.3) and (4.4) we find that

$$\begin{aligned} y_1^*[\vartheta] &= y_1[\vartheta] + \Delta y \\ y_2^*[\vartheta] &= y_2[\vartheta] - \Delta y \end{aligned} \tag{4.5}$$

$$\Delta y = -\int_\tau^{\tau+\delta} Y[\vartheta, t]\, B(t)\, \Delta u(t)\, dt \tag{4.6}$$

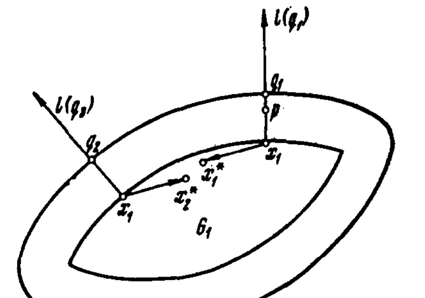

Fig. 2

We set

$$\{y_i[\vartheta]\}_m = x_i \quad \{y_i^*[\vartheta]\}_m = x_i^*$$
$$\{\Delta y\}_m = \Delta x \qquad (i = 1, 2)$$

We note that the point x_i $(i = 1, 2)$ is the point of the set $G_1[y, \tau, \vartheta]$ closest to q_i $(i = 1, 2)$. From this we obtain Eqs. (Fig. 2)

$$q_i - x_i = \varepsilon l(q_i) \qquad (i = 1, 2) \tag{4.7}$$

$$\max\,(l(q_i), x) = (l(q_i), x_i) = \mu_i - \varepsilon \quad (x \in G_1[y, \tau, \vartheta]) \tag{4.8}$$

By the definition of the attainability domain we have $x_i^* \in G_1[y, \tau, \vartheta]$, so that from (4.8) we have $(l(q_i), x_i^*) \leqslant \mu(q_i) - \varepsilon$ $(i = 1, 2)$. From (4.5) and (4.8) we have

$$(l(q_1), \Delta x) \leqslant 0 \tag{4.9}$$

$$(l(q_2), \Delta x) \geqslant 0 \tag{4.10}$$

Let $\Delta l = l(q_2) - l(q_1)$. From (4.9), (4.10) we find that $(\Delta l, \Delta x) \leqslant (l(q_1), \Delta x) \leqslant 0$; hence,

$$|(l(q_1), \Delta x)| \leqslant \|\Delta l\| \cdot \|\Delta x\| = \varphi \|\Delta x\| \tag{4.11}$$

Let us denote the hyperplane $(l(q_1), x) = 0$ by L. The vector $l(q_1)$ and the hyperplane L form an orthogonal expansion of the space E_m. Let g_1 and g_2 be the projections of the vector Δx on $l(q_1)$ and L, respectively. Then

$$\| g_2 \|^2 = \| \Delta x \|^2 - \| g_1 \|^2 \tag{4.12}$$

$$g_1 = (l(q_1), \ \Delta x)\, l(q_1), \qquad \| g_1 \| = |(l(q_1), \ \Delta x)| \leqslant \varphi \, \| \Delta x \| \tag{4.13}$$

Let us estimate the distance between q_1 and x_1^*. From (4.5), (4.7), (4.12), (4.13) we have

$$\| q_1 - x_1^* \|^2 = \| \varepsilon l(q_1) + x_1 - x_1 - \Delta x \|^2 =$$

$$= \| \varepsilon l(q_1) - g_1 - g_2 \|^2 = \| \varepsilon l(q_1) - g_1 \|^2 + \| \Delta x \|^2 - \| g_1 \|^2$$

Choosing a sufficiently small $\| \Delta x \| / \varepsilon$, we find with allowance for (4.13) that

$$\| q_1 - x_1^* \| \leqslant \varepsilon \left[\left(1 + \frac{\| g_1 \|}{\varepsilon}\right)^2 + \left(\frac{\| \Delta x \|}{\varepsilon}\right)^2 - \left(\frac{\| g_1 \|}{\varepsilon}\right)^2 \right]^{1/2} =$$

$$= \varepsilon \left(1 + \frac{\| g_1 \|}{\varepsilon}\right) + o\left(\frac{\| \Delta x \|}{\varepsilon}\right) \leqslant \varepsilon \left(1 + \frac{\varphi \| \Delta x \|}{\varepsilon}\right) + o\left(\frac{\| \Delta x \|}{\varepsilon}\right) \tag{4.14}$$

It is now easy to obtain the required estimate for the quantity $\Delta \varepsilon$. To this end we note that the point x_1^* belongs by construction to the domain $G_1^{\varepsilon^*} [y_2 [\tau + \delta], \tau + \delta, \vartheta]$. Since the distance from q_1 to this point is estimated by inequality (4.14), it follows that

$$\Delta \varepsilon = \varepsilon^* - \varepsilon \leqslant \varepsilon \left(1 + \varphi \frac{\| \Delta x \|}{\varepsilon}\right) + o\left(\frac{\| \Delta x \|}{\varepsilon}\right) - \varepsilon = \varphi \| \Delta x \| + o\left(\frac{\| \Delta x \|}{\varepsilon}\right) \tag{4.15}$$

Setting $\varepsilon \geqslant b$, where b is a fixed positive number, and recalling (4.6), we find from (4.15) that

$$\Delta \varepsilon \leqslant k \varphi \delta + o(\delta) \tag{4.16}$$

where k is some positive number. We note that all points p of the form

$$p = q_1 - \| q_1 - p \| \, l(q_1) \quad \text{for} \quad \| q_1 - p \| \leqslant \varepsilon$$

also belong to the domain $G_1^{\varepsilon + \Delta \varepsilon} [y_2 [\tau + \delta], \tau + \delta, \vartheta]$ for a $\Delta \varepsilon$ satisfying (4.16) (Fig. 2).

5. Our proof of Theorem 2.1 is based on the investigation of the variation of the quantity ε° along the trajectories of systems (1.1) and (1.2).

The quantity ε° computed at each instant $t = \tau$ from the realized $y[\tau]$ and $z[\tau]$ can be regarded as some function of time $\varepsilon^\circ[\tau] = \varepsilon^\circ [y[\tau], z[\tau], \tau, \vartheta_k]$, where a specific realization $\varepsilon^\circ[\tau]$ is associated with particular controls u and v.

We can show that in each interval $[\tau_k, \tau_{k+1})$ in the case

$$\gamma[t] = \{y[t], z[t], t, \vartheta_k\} \in \Gamma_{a,b}, \ \tau_k \leqslant t \leqslant \tau_{k+1}$$

for any $v \in V$ the selection of the control u_δ^* ensures the inequality

$$\varepsilon^\circ[\tau_{k+1}] - \varepsilon^\circ[\tau_k] \leqslant \lambda(\delta) \cdot \delta \tag{5.1}$$

Here

$$\lambda(\delta) \to 0 \quad \text{as} \quad \delta \to 0 \tag{5.2}$$

uniformly over γ from $\Gamma_{a,b}$.

We assume that $\vartheta_k = \vartheta_{k+1} = \vartheta$; otherwise (3.1) and the definitions of the instant of absorption ϑ° and the quantity ε° imply Eq. $\varepsilon^\circ[\tau_{k+1}] = 0$, which in turn implies (5.1).

Let the values of the phase vectors $y[\tau_k]$ and $z[\tau_k]$ realized at the instant $t = \tau_k$ define the attainability domains $G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_k], \tau_k, \vartheta]$ and $G_2[z[\tau_k], \tau_k, \vartheta]$. By the definition of the quantity $\varepsilon^\circ[\tau_k]$ we have

$$G_2[z[\tau_k], \tau_k, \vartheta] \subset G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_k], \tau_k, \vartheta]$$

By the instant $t = \tau_{k+1} = \tau_k + \delta$ the control u_δ^* brings system (1.1) to the state $y[\tau_{k+1}]$, and the control $v \in V$ brings system (1.2) to the state $z[\tau_{k+1}]$. The inclusion

$$G_2[z[\tau_{k+1}], \tau_{k+1}, \vartheta] \subset G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_{k+1}], \tau_{k+1}, \vartheta]$$

generally does not hold, so that we need a new and generally larger value of $\varepsilon^\circ[\tau_{k+1}]$, which will ensure the ε-absorption of process (1.2), (1.4) by process (1.1), (1.3) at the instant $t = \tau_{k+1}$. Let us obtain an upper estimate of the variation of the quantity ε°. Here we proceed on the basis of the following statement.

For any $\delta > 0$ there exists a $\zeta(\delta)$ dependent solely on δ such that for any permissable control $u(t)$ which brings system (1.1) from $y[\tau]$ to $y[\tau+\delta]$, we have

$$\rho\{g, G_1^\varepsilon[y[\tau+\delta], \tau+\delta, \vartheta] \leqslant \zeta(\delta)$$

$$\zeta(\delta) \to 0, \quad \delta \to 0, \varepsilon \geqslant 0, \quad \tau+\delta < \vartheta \tag{5.3}$$

Here g is an arbitrary point from $G_1[y[\tau], \tau, \vartheta]$, $\{y[\tau], \tau, \vartheta\}$ belongs to any bounded domain in E_{n+2}. The validity of this statement follows from the form of system (1.1) and from the character of conditions (1.3).

Let us choose a sufficiently small number $\delta > 0$ such that $\varsigma(\delta) \leqslant \min\{\alpha^\circ, b\}$; next, we set

$$\alpha = \alpha(\delta) = \zeta(\delta) \tag{5.4}$$

and find the corresponding number $\beta(\alpha) > 0$ by virtue of condition A.

Let p be an arbitrary point of the domain $G_2[z[\tau_k], \tau_k, \vartheta]$. Two cases are possible,

(1) $$S_\alpha(p) \subset G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_k], \tau_k, \vartheta] \tag{5.5}$$

(2) $$S_\alpha(p) \overline{\subset} G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_k], \tau_k, \vartheta] \tag{5.6}$$

where $S_\alpha(p)$ is a closed sphere in E_m of radius α with its center at the point p.

Let us consider the first case. We assume that

$$p \overline{\in} G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_{k+1}], \tau_{k+1}, \vartheta] \tag{5.7}$$

and set

$$g = p + \frac{\alpha(p-g^*)}{\|p-g^*\|}$$

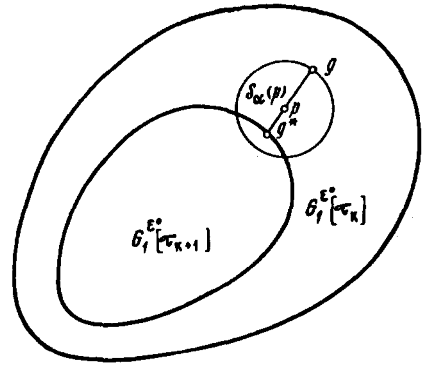

Fig. 3

Here g^* is the point of the domain $G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_{k+1}], \tau_{k+1}, \vartheta]$ nearest to p (Fig. 3). By virtue of (5.5),

$$g \in G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_k], \tau_k, \vartheta]$$

We can show that

$$\rho\{g, G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_{k+1}], \tau_{k+1}, \vartheta] = \|g-g^*\| = \alpha + \|p-g^*\| > \alpha = \zeta(\delta)$$

The latter inequality contradicts (5.3), so that assumption (5.7) is invalid, i.e. in the first case we have

$$p \in G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_{k+1}], \tau_{k+1}, \vartheta] \tag{5.8}$$

Let us consider the second case. By (5.6) there exists a point q belonging to the boundary of the domain $G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_k] \tau_k, \vartheta]$ such that $\|q-p\| \leqslant \alpha$. Let q^* be the point of the boundary of $G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_k], \tau_k, \vartheta]$ nearest to p; since $\|q^*-p\| \leqslant \alpha$, then $q^* \in N_\beta(l^\circ)$. We can show that

$$p = q^* - l(q^*)\|q^* - p\| \tag{5.9}$$

We note now that in the time interval $[\tau_k, \tau_{k+1})$ system (1.1) is subject to the control

u_δ^* which brings system (1.1) by the instant $t = \vartheta$ into the $\varepsilon^\circ[\tau_k]$-neighborhood of some point $q^\circ[\tau_k]$ belonging to the set $K[y[\tau_k], z[\tau_k], \tau_k, \vartheta]$, where $q^* \in N_\beta(l^\circ)$ and $q^\circ[\tau_k] \in N_\beta(l^\circ)$ (see (2.1)), so that $\| l(q^*) - l(q^\circ[\tau_k]) \| \leqslant 2\beta$. This means (as noted in the solution of Problem 4.1) that the points p defined by a relation of the form (5.9) for $\| q^* - p \| \leqslant \alpha \leqslant b \leqslant \varepsilon^\circ[\tau_k]$ belong to the domain

$$G_1^{\varepsilon^\circ[\tau_k]+\Delta\varepsilon}[y[\tau_{k+1}], \tau_{k+1}, \vartheta]$$

where

$$\Delta\varepsilon \leqslant 2k\beta\delta + o(\delta) \tag{5.10}$$

Recalling (5.8), we can now assert that

$$G_2[z[\tau_k], \tau_k, \vartheta] \subset G_1^{\varepsilon^\circ[\tau_k]+\Delta\varepsilon}[y[\tau_{k+1}], \tau_{k+1}, \vartheta] \tag{5.11}$$

By the definition of the attainability domain,

$$G_2[z[\tau_{k+1}], \tau_{k+1}, \vartheta] \subset G_2[z[\tau_k], \tau_k, \vartheta] \tag{5.12}$$

Inclusion (5.11) therefore implies the inequality

$$\varepsilon^\circ[\tau_{k+1}] - \varepsilon^\circ[\tau_k] \leqslant 2k\beta\delta + o(\delta) \tag{5.13}$$

Setting

$$2k\beta\delta + o(\delta) = \lambda(\delta)\cdot\delta,$$

in (5.13), we find from (5.3), (5.4) and form condition A that $\lambda(\delta) \to 0$ as $\delta \to 0$ uniformly in γ from $\Gamma_{a,b}$; this and (5.10), (5.13) imply the validity of (5.1), (5.2).

We assume now that in some interval $[\tau_k, \tau_{k+1}]$ there exists a point t_* such that $\gamma[t_*] \overline{\in} \Gamma_{a,b}$.

Let us estimate the quantity $\Delta\varepsilon^\circ = \varepsilon^\circ[\tau_{k+1}] - \varepsilon^\circ[\tau_k]$ in this case. Since we are limiting ourselves to the upper estimate of the quantity $\Delta\varepsilon^\circ$, we again assume that $\vartheta_k = \vartheta_{k+1} = \vartheta$.

To find the required estimate we make use of relation (5.3), from which we find that

$$G_1^{\varepsilon^\circ[\tau_k]}[y[\tau_k], \tau_k, \vartheta] \subset G_1^{\varepsilon^\circ[\tau_k]+\Delta\varepsilon}[y[\tau_{k+1}], \tau_{k+1}, \vartheta] \qquad (\Delta\varepsilon \leqslant \zeta(\delta))$$

The inclusion (5.12) implies in this case that

$$\Delta\varepsilon^\circ = \varepsilon^\circ[\tau_{k+1}] - \varepsilon^\circ[\tau_k] \leqslant \zeta(\delta) \tag{5.14}$$

We shall now formulate our last ancillary statement.

The attainability domain $G_1[y, \tau, \vartheta]$ belongs to some sphere S_ρ of radius ρ and

$$\rho(a) \to 0 \quad \text{for } a = \vartheta - \tau \to 0 \tag{5.15}$$

monotonically and uniformly in all $\{y, \tau, \vartheta\}$ from any bounded domain.

The validity of this statement follows from the form of system (1.1) and from the character of restrictions (1.3).

Finally, let us show that a given number $\eta > 0$ can be used to find a $\delta^\circ > 0$ such that (2.2) holds. We choose the numbers $a > 0$, $b > 0$ such that

$$2b + \rho(a) \leqslant {}^1/_4\,\eta \tag{5.16}$$

This is also possible by virtue of (5.15). The numbers a and b in turn determine the domain $\Gamma_{a,b}$.

We assume now that τ_* is the instant when the inequality $\vartheta_i - \tau \geqslant a$ $(\tau_i \leqslant \tau < < \tau_{i+1})$ is first violated. (By construction of the numbers ϑ_i such an instant necessarily arrives). Two cases are possible,

$$(1)\ \varepsilon^\circ [y[\tau_*], z[\tau_*], \tau_*, \vartheta_i] = \varepsilon^\circ [\tau_*] < b$$
$$(2)\ \varepsilon^\circ [y[\tau_*], z[\tau_*], \tau_*, \vartheta_i] = \varepsilon^\circ [\tau_*] \geqslant b$$

Let us consider the first case. By the definition of the quantity $\varepsilon^\circ [\tau_*]$ we have

$$G_2 [z[\tau_*], \tau_*, \vartheta_i] \subset G_1^{\varepsilon^\circ [\tau_*]} [y[\tau_*], \tau_*, \vartheta_i] \tag{5.17}$$

As noted above, the domain $G_1 [y[\tau_*], \tau_*, \vartheta_i]$ belongs to some sphere of the radius $\rho(\vartheta_i - \tau_*) \leqslant \rho(a)$, so that the domain $G_1^{\varepsilon^\circ [\tau_*]}[y[\tau_*], \tau_*, \vartheta_i]$ lies in a sphere of radius

$$r = \rho(a) + 2\varepsilon^\circ [\tau_*] < \rho(a) + 2b \leqslant {}^1/_4\, \eta$$

By virtue of (5.17) $\{y[\vartheta_i]\}_m$ and $\{z[\vartheta_i]\}_m$ lie inside a sphere of radius $r \leqslant {}^1/_4\, \eta$ at the instant ϑ_i for any controls u and v; this implies that $\|\{y[\vartheta_i] - z[\vartheta_i]\}_m\| \leqslant \leqslant 2\ \eta/4 = {}^1/_2\eta$; here, by virtue of (3.1), $\vartheta_i \leqslant \vartheta^\circ$, so that in the first case we have (2.5).

Let us consider the second case. Let τ_{**} be the last instant when

$$\varepsilon^\circ [y[\tau], z[\tau], \tau, \vartheta_j] = b, \ \tau_j \leqslant \tau < \tau_{j+1}.$$

From (5.14) we find that

$$\varepsilon^\circ [\tau_{j+1}] \leqslant b + \zeta(\delta)$$

Here, beginning at the instant τ_{**} and ending at the instant τ_*, the vector $\gamma[t] \in \Gamma_{a,b}$; hence, estimates (5.1) and (5.2) apply from the instant τ_{j+1} to the instant τ_*; from this, with allowance for the inequality $\tau_* - \tau_{j+1} \leqslant \vartheta^\circ - t^\circ$ we obtain

$$\varepsilon^\circ [\tau_*] \leqslant b + \zeta[\delta] + \lambda(\delta)(\vartheta^\circ - t_0)$$

As in the first case, this implies that the points $\{y[\vartheta_i]\}_m$ and $\{z[\vartheta_i]\}_m$ lie in a sphere of radius

$$r = \rho(a) + 2\varepsilon^\circ [\tau_*] \leqslant \rho(a) + 2b + 2(\zeta(\delta) + \lambda(\delta)(\vartheta^\circ - t_0))$$

By virtue of (5.2), (5.3), there exists a $\delta^\circ > 0$ such that

$$2(\zeta(\delta) + \lambda(\delta)(\vartheta^\circ - t_0)) \leqslant {}^1/_4\eta \quad \text{for } 0 < \delta \leqslant \delta^\circ$$

With the number δ° chosen in this way we have $r \leqslant {}^1/_2\eta$ so that

$$\|\{y[\vartheta_i] - z[\vartheta_i]\}_m\| \leqslant \eta \qquad (\vartheta_i \leqslant \vartheta^\circ)$$

Hence, Theorem 2.1 has been proved.

6. In proving Theorem 2.1 we showed that the control u_δ^* of the form (1.6) whose construction is described in Section 3 ensures fulfillment of relations (5.1), (5.2). This control is some vector function of time in each interval $[\tau_k, \tau_{k+1})$ $(k = 0, 1, \ldots)$. We can show now that among the controls u_δ of the form

$$u_\delta = u_\delta [y[\tau_k], z[\tau_k], \tau_k, \vartheta_k] \tag{6.1}$$

i.e. among the controls constant over each semi-interval $[\tau_k, \tau_{k+1})$ there exists a permissible control u_δ° given by Eq.

$$u_\delta^\circ = \frac{1}{\delta} \int_{\tau_k}^{\tau_{k+1}} u_\delta^* [t, y[\tau_k], z[\tau_k], \tau_k, \vartheta_k]\, dt \tag{6.2}$$

which also ensures fulfillment of relations (5.1), (5.2).

To this end, making use of the Cauchy formula, we obtain the inequality

$$\|y^* [\tau_{k+1}] - y^\circ [\tau_{k+1}]\| \leqslant \sigma (\delta) \cdot \delta \tag{6.3}$$

where $y^* [\tau_{k+1}]$, $y^\circ [\tau_{k+1}]$ are the states to which the controls $u_\delta{}^*$ and $u_\delta{}^\circ$ bring system (1.1) from the state $y = y [\tau_k]$; the function σ (δ) satisfies the condition

$$\sigma (\delta) \to 0 \quad \text{as} \quad \delta \to 0 \tag{6.4}$$

uniformly in every domain $\Gamma_{a,b}$.

We readily infer from (6.3), (6.4) that relations (5.1), (5.2) remain valid for $u = u_\delta{}^\circ$ and for all $v \in V$. As in proving Theorem 2.1, we can now verify the validity of the following statement.

Theorem 6.1. If condition A is fulfilled, then a control $u_\delta{}^\circ$ of the form (6.1) ensures convergence of the motions $y [t]$ and $z [t]$ not later than at the instant $t = \vartheta^\circ$

For example, let us consider the problem of encounter of two material points of unit mass M_1 and M_2 moving in a vertical plane. Their equations of motion are

$$y_1{}^\cdot = y_3, \qquad y_2{}^\cdot = y_4, \qquad y_3{}^\cdot = u_1, \qquad y_4{}^\cdot = u_2 - g \tag{6.5}$$

$$z_1{}^\cdot = z_2, \qquad z_2{}^\cdot = z_4, \qquad z_3{}^\cdot = v_1, \qquad z_4{}^\cdot = v_2 - g \tag{6.6}$$

where y_1, y_2 and z_1, z_2 are the coordinates of the pursuing and pursued points, respectively; y_3, y_4 and z_3, z_4 are the components of the velocities of the pursuing and pursued objects; g is the gravitational acceleration; the controls $u = \{u_1, u_2\}$ and $v = \{v_1, v_2\}$ are restricted by conditions of the form

$$u_1{}^2 + u_2{}^2 \leqslant \mu^2, \qquad v_1{}^2 + v_2{}^2 \leqslant \nu^2, \qquad \mu > \nu \tag{6.7}$$

By the "encounter" of objects (6.5) and (6.6) we mean the coincidence of the coordinates of the points M_1 and M_2.

The attainability domains $G_1 [y, \tau, \vartheta]$ and $G_2 [z, \tau, \vartheta]$ constructed in the plane g_1, g_2 are the disks

$$[g_1 - (y_1 + Ty_3)]^2 + [g_2 - (y_2 + Ty_4 - {}^1/_2 gT^2)]^2 \leqslant R^2 \tag{6.8}$$

$$[g_1 - (z_1 + Tz_3)]^2 + [g_2 - (z_2 + Tz_4 - {}^1/_2 gT^2]^2 \leqslant r^2 \tag{6.9}$$

whose radii are $R = {}^1/_2 \mu T^2$ and $r = {}^1/_2 \nu T^2$, where $T = \vartheta - \tau$. From (6.8), (6.9) we readily obtain the following equation for determining the instant of absorption $\vartheta^\circ [y, z, \tau]$:

$${}^1/_4 (\mu - \nu)^2 (\vartheta - \tau)^4 - [x_1 + x_3 (\vartheta - \tau)]^2 - [x_2 + x_4 (\vartheta - \tau)]^2 = 0 \tag{6.10}$$

$$x_i = y_i - z_i \qquad (i = 1, 2, 3, 4)$$

The quantity $\vartheta^\circ [y, z, \tau]$ is the smallest positive root of this equation.

Since $R > r$ for $\mu > \nu$ and $T = \vartheta - \tau > 0$, the boundaries of the domains $G_1{}^{\varepsilon^\circ}$ and G_2 always touch at a single point q^0, so that condition A is fulfilled in this example.

Fig. 4 shows some computer-simulated realizations of the pursuit process for $\mu = 60$, $\nu = 60 - 10\sqrt{5}$, $g = 10$. At the initial instant $t_0 = 0$ the objects are in the states

$$y_1 (0) = y_2 (0) = y_3 (0) = y_4 (0) = 0$$

$$z_1 (0) = 0, \quad z_2 (0) = 15, \quad z_3 (0) = 5, \quad z_4 (0) = -5$$

The solid curves in Fig. 4 represent the trajectories of objects (6.5), (6.6) in the case where the pursuer employes the control u_δ° and the pursued (target) employs the extremal control v_e, i.e. the control which at each instant aims the motion of system (6.6) at the point of tangency of the attainability domain boundaries. Encounter in this case occurs at the instant $t = \vartheta^\circ [t_0] = 1$.

The dot curves represent the trajectories of the pursued object in the case where the pursuer employs the control $u_\delta{}^\circ$, while the target deviates from the extremal strategy.

Thus, the ascending trajectories correspond to the case where the target directs the force v of magnitude ν to one side of the pursuer throughout the process; encounter in this case occurs at the instant $t = 0.97 < \vartheta^\circ[t_0] = 1$. The trajectories proceeding towards the left are realized when the target chooses $v = \{-\nu, 0\}$ throughout the process; encounter in this case occurs at the instant $t = 0.73 < \vartheta^\circ[t_0] = 1$.

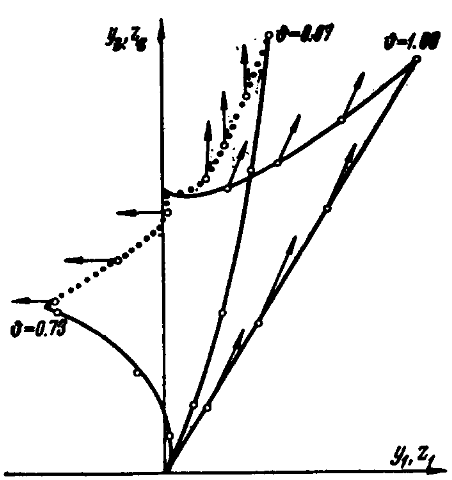

Fig. 4

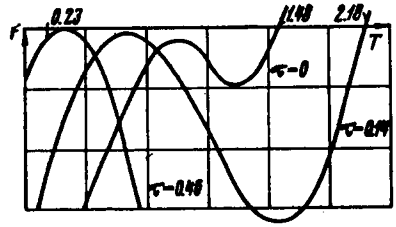

Fig. 5

The inequality $\vartheta_k \leqslant \vartheta_{k-1}$ is fulfilled in each interval of the pursuit process realizations considered, so that the control u_8° coincides with the extremal control.

We note that in the above example $T^\circ = \vartheta^\circ[t_0] - t_0$ is the minimax of the time-to-encounter, and that the extremal control u_e solves the problem of the minimax of the time-to-encounter of the motions $y[t]$ and $z[t]$, although the pair of extremal controls u_e, v_e does not yield the saddle point of the game (as is evident from the example).

At the initial instant $t = 0$ let

$$y_1(0) = -3/2\sqrt{2}, \quad y_2(0) = {}^1/_2\sqrt{2}, \quad y_3(0) = \sqrt{2}, \quad y_4(0) = 0.1/8\sqrt{2}$$

$$z_1(0) = z_2(0) = z_3(0) = z_4(0) = 0$$

We set $\mu = 1.5$, $\nu = 0.5$, $g = 0$. Fig. 5 shows plots of

$$F = F(\tau, T) = {}^1/_4(\mu - \nu)^2 T^4 - \{x_1[\tau] + Tx_3[\tau]\}^2 - \{x_2[\tau] + Tx_4[\tau]\}^2$$

for several values of τ for $v = v_e$ and $u = \{0, -\mu\}$ which is not extremal.

From the process of deformation of the curve $F = F(\tau, T)$ we see that from the instant $\tau = t_0 = 0$ to the instant $\tau_* = 0.46$ the smallest positive root of Eq. (6.10) increases, although at the instant $\tau_* = 0.46$ Eq. (6.1) has a new root $T = \vartheta - \tau = 0.23$; by changing over to the extremal control at this instant, the pursuer ensures encounter not later than at the instant $\vartheta = 0.46 + 0.23 = 0.69$ (i.e. much sooner than at the instant $\vartheta^\circ[t_0] = 1.48$) for any permissible control $v[t]$ for $t \geqslant \tau_*$.

We have shown that the inequality $T_{u, v_e} \geqslant T_{u_e, v_e}$ is invalid in this case, so that the pair u_e, v_e does not yield the saddle point of the game under consideration.

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